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Comparative study of the Saturated Sliding Mode and LQR Controllers

Chaker Zaafouri
Research Unit c3s ESSTT,
University of Tunis
TUNISIA
chaker.zaafouri@enib.rnu.tn

Anis Sellami
Research Unit c3s ESSTT,
University of Tunis
TUNISIA
anis.sellami@esstt.rnu.tn

Borhen Torchani
Research Unit c3s ESSTT,
University of Tunis
TUNISIA
bourhen@gmail.com

Germain Garcia
LAAS-CNRS
University of Toulouse
FRANCE
Garcia@laas.fr

***Abstract*—The saturation problem is the one of the most common handicaps for applying linear control to real applications, especially the actuator saturation. This paper focus on a comparative study between the classic Linear Quadratic Regulator (LQR) control and robust saturated sliding mode control. In the first step, we present a design methodology of SMC of a class of linear saturated systems. We introduce the structure of the saturation, them we perform the design of the sliding surface as a problem of root clustering, which leads to the development of a smooth and non-linear control law that ensures to reach the sliding surface. The second step is devoted to present briefly the Linear Quadratic Regulator (LQR) control technique.The constraint of saturation is reported on the control vector. To highlight results we present a comparative analysis with a SMC and LQR controllers with saturation. Finally, we use an example of a quarter of vehicle system to give simulation results.**

Keywords: Variable Structure Control; Sliding Mode Control; LQR regulator; Saturation; Robustness; LMI.

I. Introduction

The problem of saturation remains one of the obstacles to provide properties of guarantee on the stability of systems. Used in early days ([1], [2]...), and many other methods which introduce conditions on systems containing saturation functions ([3], [4], [5]...). Most industrial processes operate in the areas caracterised by many physical and technological constraints (saturation, limit switches...). The implementation of the control law designed without considering these limitations can have dire consequences for the system. The problem of the control of saturated systems is a subject of great interest for applications. Used in early days, many rigorous design methods are available to provide guarantee properties on systems stability. In robustness terms the sliding mode is a very significant transitory mode for the Variable Structure Control (VSC), ([6], [7]...). Early work was mainly done by Soviet control scientists ([8],[9]...). In recent years, we find more research and many successful applications ([10], [11], [12]...). This paper is organized as follows: in the beginning, we give a short introduction on the structure of the saturation constraint reported on the control vector and its implementation in the system. We will then present a design procedure of robust saturated sliding mode control.This controller development procedure contains the classical steps of sliding mode design. The first one is to build an optimal sliding surface.The choice of the sliding surface is formulated as a pole assignment of a reduced order linear uncertain system in a regin through convex optimization The solution to this problem is therefore numerically tractable via standard LMI optimization and the existing robust linear system theory , and the second one is to choose a control law to enforce the system behavior to reach and stay in the desired sliding surface. To validate the theoretical concepts of this work, we treated an application of a quarter of vehicle system where we will highlight a comparison between the sliding mode control and the LQR controllers.

II. System with saturation constraint

Let us consider that the structure of the saturation constraint is described by figure 1:

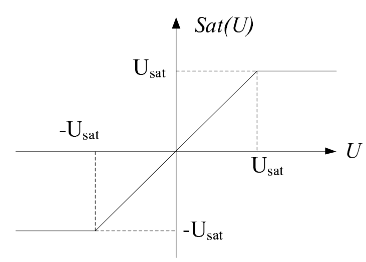

Fig. 1. The Structure of the saturation constraint

Assumption 1: The control vector is subjected to constant limitations in amplitude. It's defined by:

$$u \in \Re^m = \{u \in \Re^m / -U_{sat} \leq u_i \leq U_{sat}; U_{sat} > 0\} \quad (1)$$

The term of saturation has the following form

$$sat(u) = \begin{cases} U_{sat} & \text{if} \quad u_i > U_{sat} \\ u_i & \text{if} \; -U_{sat} < u_i < U_{sat} \\ -U_{sat} & \text{if} \quad u_i < -U_{sat} \end{cases}, \; \forall i = 1, ..., m \quad (2)$$

we can write

$$sat(u) = \beta u \quad (3)$$

the elements of β_i are expressed as follows

$$\beta_i = \begin{cases} \frac{U_{sat}}{u_i} & \text{if} \quad u_i > U_{sat} \\ 1 & \text{if} \ -U_{sat} < u_i < U_{sat} \\ \frac{-U_{sat}}{u_i} & \text{if} \quad u_i < -U_{sat} \end{cases}, \ \forall i = 1, ..., m \tag{4}$$

The saturated system can be written as:

$$\dot{x}(t) = Ax(t) + B\beta u(t) \tag{5}$$

with $A \in \Re^{n\times n}, B \in \Re^{n\times m}, x \in \Re^n$ and $u \in \Re^m$

Assumption 2: The pair (A, B) is controllable,B has full rank m, and $n > m$.

III. Sliding Mode Control (SMC)

The sliding mode occurs when the state reaches and remains in the surface given by :

$$S = \bigcap_{j=1}^{m} S_j = \{x \in \Re^n / Cx = 0\} \tag{6}$$

The sliding mode occurs when the state reaches and remains in the surface intersection S of the m hyperplanes, geometrically the subspace S is the null space of C.

Differentiating with respect the time,

$$\dot{s} = CAx + CB\beta u = 0 \tag{7}$$

if $(CB\beta)^{-1}$ exists, then

$$u_{eq} = -(CB\beta)^{-1} CAx = -Kx \tag{8}$$

with $K = (CB\beta)^{-1}CA$

$$\dot{x} = (I_n - \beta B (CB\beta)^{-1} C)Ax = A_{eq}x \tag{9}$$

the dynamics $\dot{x}$ (equation 9) describes the motion on the sliding surface and depends only on the choice of C.

1) Design of the sliding surface: The canonical form used in Reference [4] for VSC design to select the gain matrix C that gives a good and stable motion during the sliding mode.

By assumption, the matrix B has full rank m; as a result, there exists an (n x n) orthogonal transformation matrix T such that: $TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$, where B_2 is (m x m) and non-singular. Note that the choice of an orthogonal matrix T avoids inverting T when transforming back to the original system. As the transformed state variable vector is defined as:

$$\mathrm{y} = Tx \tag{10}$$

the state equation becomes

$$\dot{\mathrm{y}}(\mathrm{t}) = \mathrm{T}x(\mathrm{t}) = \mathrm{T}Ax(t) + TB\beta u(t) \tag{11}$$

If the transformed state is partitioned as $y^T = \begin{bmatrix} y_1^T & y_2^T \end{bmatrix}$; $y_1 \in \Re^{n-m}$; $y_2 \in \Re^m$

then

$$\begin{cases} \dot{y}_1(t) = A_{11}y_1(t) + A_{12}y_2(t) \\ \dot{y}_2(t) = A_{21}y_1(t) + A_{22}y_2(t) + \beta B_2 u(t) \end{cases} \tag{12}$$

Since the sliding condition is

$$Cx = CT^T y = 0 \tag{13}$$

witch

$$\mathrm{TAT}^T = \begin{bmatrix} A_{11} A_{12} \\ A_{21} A_{22} \end{bmatrix}, \mathrm{CT}^T = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{14}$$

We can the new defining sliding condition

$$C_1 y_1 + C_2 y_2 = 0 \tag{15}$$

Assumption 3: CB is non-singular then C_2 must be non-singular.

The sliding mode condition becomes

$$y_2 = -C_2^{-1} C_1 y_1 = -F y_1 \tag{16}$$

wich $F = C_2^{-1} C_1$ being an [m x (n-m)] matrix. The sliding mode is then governed by the equations

$$\begin{cases} \dot{y}_1 = A_{11}y_1 + A_{12}y_2 \\ y_2 = -Fy_1 \end{cases} \tag{17}$$

representing an $(n-m)^{th}$ order system with y_2 playing the role of a state feedback control.The closed-loop system will then have the dynamics $\dot{y}_1 = (A_{11} - A_{12}F)y_1$ This indicates that the design of a stable sliding mode requires the selection of a matrix F such that $\dot{y}_1 = (A_{11} - A_{12}F)y_1$ has $(n-m)$ left half-plane eigenvalues. Performances are taken into account via root clustering of the closed-loop dynamic matrix in a region of the complex plane. The area $\Omega(\alpha, -q, r, \theta)$ considered here is defined in Figure 2,which ensures a minimum decay rate $\alpha < 0$, a minimum damping ratio $\xi = \cos\theta$, and for relative stability and speed limitation can be made to place the eigenvalues in a circle in the left half complex plane.

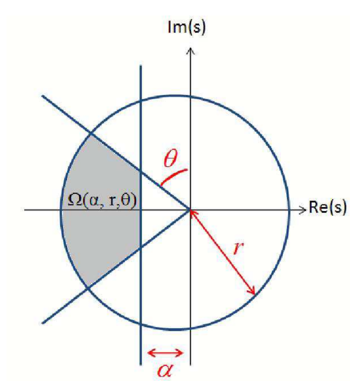

Fig. 2. LMI Region:intersection of three elementary regions.

Where F must be selected, that those (n-m) eigenvalues of the system are in the region $\Omega(\alpha, -q, r, \theta)$, it can be determined by using root clustering with LMI concept. C is given by them :

$$C = [F \quad \mathrm{I}_m]\, T \tag{18}$$

2) Saturated control law design: Once the existence problem has been solved that is the matrix C has been determined, attention must be turned to solving the reachability problem. This involves the selection of a feedback control function $u(x)$ which ensures that trajectories are directed towards the switching surface from any point in the state space. The control strategy used here will be derived from that of Reference [13] which originated from the work of Gutman [14], and it consists of the sum of a linear control law u^L and a nonlinear part u^N. The general form is:

$$u(x) = u^L(x) + u^N(x) = Lx + \rho \frac{Nx}{\|Mx\| + \delta} \tag{19}$$

where L is an (n-m) matrix, the null spaces of the matrices N; M; and C are coincident, and δ is a small positive constant to replace the discontinuous component by a smooth nonlinear function, yielding chattering-free system response.

Starting from the transformed state y, we form a second transformation $T_2 : \Re^n \to \Re^n$ such that

$$Z = T_2 y = T_2 T x;\ Z^T = \left[z_1^T \ z_2^T\right] \tag{20}$$

And $z_1 \in \Re^{n-m};\ z_2 \in \Re^m$

where

$$T_2 = \left[\begin{array}{cc} I_{n-m} & 0 \\ F & I_m \end{array}\right] \tag{21}$$

T_2 is non-singular as its inverse is given by

$$T_2^{-1} = \left[\begin{array}{cc} I_{n-m} & 0 \\ -F & I_m \end{array}\right] \tag{22}$$

The new state variables are then

$$\left\{\begin{array}{l} z_1 = y_1 \\ z_2 = F y_1 + y_2 \end{array}\right. \tag{23}$$

and the transformed system equation becomes

$$\left\{\begin{array}{l} \dot{z}_1 = \sum_1 z_1 + \sum_2 z_2 \\ \dot{z}_2 = \sum_3 z_1 + \sum_4 z_2 + \beta B_2 u \end{array}\right. \tag{24}$$

with

$$\left\{\begin{array}{l} \sum_1 = A_{11} - A_{12}F \\ \sum_2 = A_{12} \\ \sum_3 = F\sum_1 - A_{22}F + A_{21} \\ \sum_4 = A_{22} + A_{12}F \end{array}\right. \tag{25}$$

In order to attain the ideal sliding mode, it is necessary to force z_2 and $\dot{z}_2$ to become identically zero. To this end, the linear control law part u^L is formulated as

$$u^L(z) = -\left(\beta B_2\right)^{-1} \left[\Sigma_3 z_1 \quad (\Sigma_4 - \Sigma_4^*) z_2\right] \tag{26}$$

where $\sum_4^* \in \Re^{m*m}$ is any design matrix with stable eigenvalues. In particular, we may set $\sum_4^* = diag\left(\mu_i\right)$ such that $\mathrm{Re}\left(\mu_i\right) < 0$ fot i = 1 to m. Transforming back into the original x-space yields

$$u^L(x) = Lx = -\left(\beta B_2\right)^{-1} \left[\Sigma_3 \quad (\Sigma_4 - \Sigma_4^*)\right] T_2 T x \tag{27}$$

$$L = -\left(\beta B_2\right)^{-1} \left[\Sigma_3 \quad (\Sigma_4 - \Sigma_4^*)\right] T_2 T \tag{28}$$

Before presenting the nonlinear control law part u^N letting the matrix P_2 denote the positive definite unique solution of the Lyapunov equation

$$P_2 \Sigma_4^* + \Sigma_4^* P_2 + \mathrm{I}_m = 0 \tag{29}$$

then $P_2 z_2 = 0$ if and only if $z_2 = 0$, and we may take

$$u^N = -\rho \frac{\left(\beta B_2\right)^{-1} P_2 z_2}{\|P_2 z_2\| + \delta} \tag{30}$$

Transforming back into the original x-space, we obtain

$$u^N = -\rho \frac{\left(\beta B_2\right)^{-1} \left[0 \quad P_2\right] T T_2 x}{\|\left[0 \quad P_2\right] T T_2 x\| + \delta} \tag{31}$$

since the existence of the nonlinear component is checked them we can deduce the matrices N and M

$$N = -\left(\beta B_2\right)^{-1} \left[0 \quad P_2\right] T T_2 \tag{32}$$

$$M = \left[0 \quad P_2\right] T T_2 \tag{33}$$

3) Invariance of the sliding mode: Let us consider a continuous linear uncertain system described by

$$\dot{x}(t) = (A + \Delta A) x(t) + (B + \Delta B) u(t) \tag{34}$$

$\Delta A \,\mathrm{and} \Delta B$ are the uncertainty of matching conditions type written as

$$\Delta A = B \Delta \tilde{A}, \Delta B = B \Delta \tilde{B} \tag{35}$$

If the system is in sliding mode then $s = Cx(t) = 0$ En différentiant par rapport au temps

$$\dot{s} = C\left(A + B\Delta\tilde{A}\right) x\left(t\right) + CB\left(I + \Delta\tilde{B}\right) u\left(t\right) = 0 \tag{36}$$

if $\left(I + \Delta\tilde{B}\right)^{-1}$ exists , then

$$u_{eq} = -\left(I + \Delta\tilde{B}\right)^{-1} \left(CB\right)^{-1} C\left(A + B\Delta\tilde{A}\right) x \tag{37}$$

As a result

$$\dot{x}\left(t\right) = \left(A + B\Delta\tilde{A}\right) x - B\left(CB\right)^{-1} C\left(A + B\Delta\tilde{A}\right) x \tag{38}$$

We finally get:

$$\dot{x}\left(t\right) = \left(I - B\left(CB\right)^{-1} C\right) A x = A_{eq} x \tag{39}$$

The dynamics $\dot{x}\left(t\right) = A_{eq} x$ describes the motion on the sliding surface which is independent of $\Delta A \,\mathrm{and} \Delta B$ and depends only on the choice of the matrix C.

IV. Linear Quadratic Regulator (LQR)

In this part we briefly recall the principle of the LQR control.
LQR: linear quadratic regulator. The system is linear and the control is quadratic. Let us consider the linear system gives in (45).

Assumption 4: The pair (A,B) is stabilisable, i.e. there is no unstable and ungovernable mode in the system.

Consider a state variable feedback regulator in the form of

$$u = -kx \tag{40}$$

The optimization procedure to obtain the values of K consists of determining the control input u, which minimizes the performance index J. The performance index J represents the performance characteristic requirement as well as the controller input limitation [15].

The state feedback control which stabilizes the system and minimizes the LQR criterion

$$J = \int_0^\infty \left(x^T Q x + u^T R u\right) dt \tag{41}$$

with R > 0 and Q $\geq$ 0

The matrix gain K is represented by;

$$k = R^{-1} B^T P \tag{42}$$

P is a definite positive solution of the equation of RICCATI:

$$PA + A^T P - PBR^{-1}B^T P + Qx = 0 \tag{43}$$

we obtain then $J_{\min} = x_0^T P x_0$ with $x(t=0) = x_0$.
Then the feedback regulator:

$$u = -(R^{-1} B^T \bar{P}) x \tag{44}$$

For the LQR control, we will use the same structure of saturation applied to the SMC presented in the first part.

V. Numerical application

We consider a two degree of freedom vibrating system with one actuator describes by figure 4 The state equation of the system is given by,[16]:

$$\dot{X}(t) = Ax(t) + Bu(t) \tag{45}$$

with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{C_1+C_2}{m_1} & \frac{C_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{C_2}{m_2} & -\frac{C_2}{m_2} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{pmatrix}$$

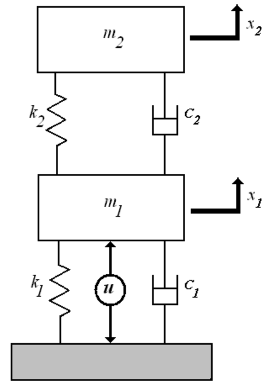

Fig. 3. Two degrees of freedom vibrating system with one actuator

Simulation is achieved under the following condition:

$$m_1 = m_2 = 1, k_1 = k_2 = 1, C_1 = C_2 = 0.01,$$
$$-1 \leq u(t) \leq 1$$

The initial condition is given by $x_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$

The figure (5) represents the poles of the reduced order system in an area defined by $\Omega(\alpha, -q, r, \theta) = \Omega(-0.4, 0, 2.5, pi/8)$ in the complex plan.

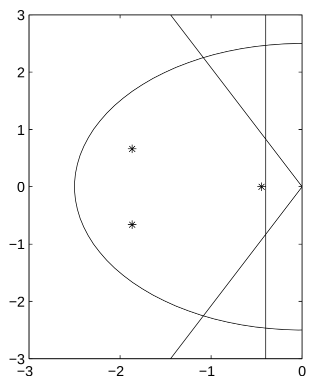

Fig. 4. Poles of the reduced system

After 3 iterations, the algorithm gives the stabilizing gain F of the reduced system

$$F = \begin{bmatrix} 4.1215 & 2.3786 & -4.5626 \end{bmatrix}$$

According to (18)

$$C = \begin{bmatrix} 4.1215 & -2.3786 & 1.0000 & 4.5626 \end{bmatrix}$$

The following figure 5 and figure 6 presents, respectively, the evolution of control input and the state variables (- - -: LQR controller, –: SMC).

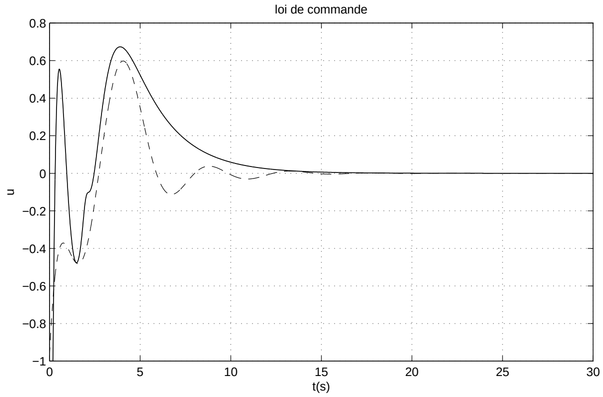

Fig. 5. Evolution of the control law (- -: LQR, –: sliding mode)

Figure (5) presents the evolution of the saturated control input. It's clear that the two controllers are saturated and always inferior to its maximal value in the two cases, but we can check that the LQR controller have a more transient mode and the convergence is more slowly than that of the first control , what proves that the robust stability of the SMC is checked.

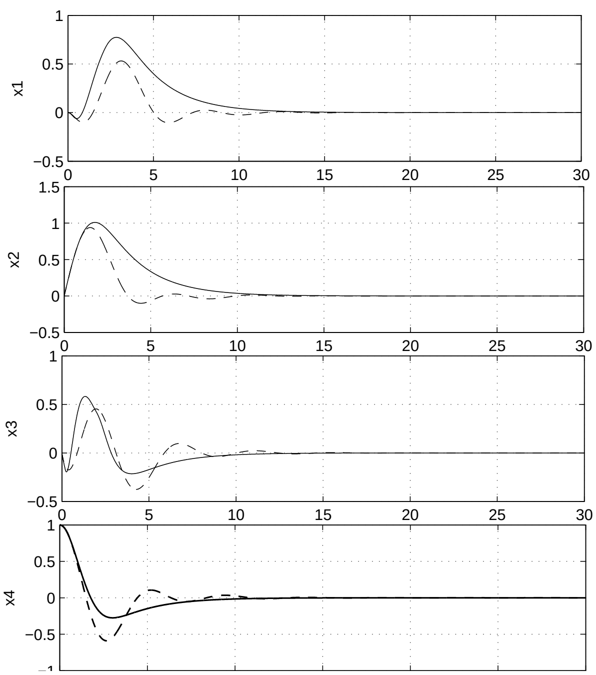

Fig. 6. System response (x1 to x4), (- -: LQR, –: sliding mode)

Figure (6) compares displacements of x1, x2, x3 and x4 with saturation. It shows a typical stable sliding mode convergence of the system using sliding mode control. As consequence, the state variables dynamics of the system with LQR controller have a more transient mode and the convergence is more slowly than that of the first control.

VI. Conclusion

In this paper, we proposed a comparative study of the sliding mode and LQR controllers for linear time invariant saturated systems. The structure of the saturation constraint is reported on the control input and being of constant limitations in amplitude. The design of the sliding surface is formulated as a pole assignment of a reduced system in an LMI region.The non-linear saturated control scheme is introduced, will be ensure the elimination of the undesirable chattering phenomenon and ensure a stable sliding mode motion. After that, we briefly had the principle and results of LQR controller and we will use the same structure of saturation presented. To verify the performance of the proposed SMC, we presented the simulation results for two controllers, applied to the "Two degrees of freedom vibrating system with one actuator". Indeed these simulation results show that the LQR controller is stable and acceptable,but the convergence is slowly. The SMC can remove the transient mode had been the main defect of the LQR control, and has the better performance. Consequently, we verified that the proposed SMC had the better robustness performance than the LQR control.

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